MATH 5061 Problem Set 4[1]
Due date: Mar 17, 2021

Problems: (Please hand in your assignments via Blackboard. **Late submissions will not be accepted.**)

Throughout this assignment, we use (M, g) to denote a smooth n-dimensional Riemannian manifold with its Levi-Civita connection ∇ unless otherwise stated. The Riemann curvature tensor (as a $(0,4)$-tensor) of (M, g) is denoted by R.

1. Prove the *second Bianchi identity*: for any vector fields $X, Y, Z, W, T \in \Gamma(TM)$,

$$(\nabla_X R)(Y, Z, W, T) + (\nabla_Y R)(Z, X, W, T) + (\nabla_Z R)(X, Y, W, T) = 0.$$

2. Suppose that (M^n, g) is a connected Riemannian manifold with $n \geq 3$ such that there exists a function $f : M \to \mathbb{R}$ such that $K(\sigma) = f(p)$ for all two-dimensional subspace $\sigma \subset T_pM$. Show that f must be a constant function on M.(*Hint: use the second Bianchi identity*)

3. A Riemannian manifold (M^n, g) is called *Einstein manifold* if there exists a smooth function $\lambda : M \to \mathbb{R}$ such that $\mathrm{Ric}(X, Y) = \lambda \langle X, Y \rangle$ for any vector fields $X, Y \in \Gamma(TM)$.

   (a) Suppose (M^n, g) is a connected Einstein manifold with $n \geq 3$, show that λ must be a constant function.

   (b) Suppose (M^3, g) is a connected 3-dimensional Einstein manifold. Show that M has constant sectional curvature.

4. Let $f : M \to \mathbb{R}$ be a smooth function defined on a Riemannian manifold (M^n, g). Denote $\Sigma := f^{-1}(a)$ where a is a regular value of f. Show that the mean curvature H, with respect to the unit normal $N = -\frac{\nabla f}{|\nabla f|}$, of the hypersurface Σ is given by $H = \pm \mathrm{div} N$ (up to a sign depending on the sign convention in the definition of mean curvature).

5. Consider the smooth map $F : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$F(u, v) = (\cos u, \sin u, \cos v, \sin v).$$

   (a) Show that F is an isometric immersion (with respect to the flat metrics).

   (b) Prove that the image of F lies inside the round 3-sphere $\mathbb{S}^3 := \{x \in \mathbb{R}^4 \mid |x|^2 = 2\}$, and $\Sigma = F(\mathbb{R}^2)$ is a minimal immersion into $\mathbb{S}^3$, equipped with the induced metric from $\mathbb{R}^4$.

[1] Last revised on March 19, 2021